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Analysis of Spectral Points of the Operators $T^{[\natural]}T$ and $TT^{[\natural]}$ in a Krein Space

André Ran and Michał Wojtylak

Abstract. Spectra and sets of regular and singular critical points of definitizable operators of the form $T^{[\natural]}T$ and $TT^{[\natural]}$ in a Krein space are compared. The relation between the Jordan chains of the above operators (corresponding to the same eigenvalue) is shown.

Mathematics Subject Classification (2000). 47B50.

Keywords. Regular critical point, singular critical point, Jordan chain, Krein space.

1. Introduction

In this paper we will deal with a pair of operators $T^{[\natural]}T$ and $TT^{[\natural]}$, where T is a possibly unbounded operator in a Krein space. If T is bounded, then by the classical result in [8] the nonzero spectra of the above operators coincide [8]. Apparently, we cannot say much on the unbounded case without assuming anything about the operator T . Our main additional condition will be that both operators $T^{[\natural]}T$ and $TT^{[\natural]}$ are selfadjoint and definitizable. The reasons for this (nontrivial) requirement can be found in Section 3. In this setting the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ divides (with respect to $T^{[\natural]}T$) into four parts. A point λ either belongs to the resolvent $\rho(T^{[\natural]}T)$, or it belongs to the definite spectrum $\sigma_+(T^{[\natural]}T) \cup \sigma_-(TT^{[\natural]})$, or it is a regular critical point, or it is a singular critical point (see Section 2 for definitions). The same decomposition can be done with respect to $TT^{[\natural]}$. Our motivation for the research presented in this paper was to compare these two divisions of $\overline{\mathbb{R}}$.

A more general problem was studied in the finite dimensional case in the paper [6] by Flanders, already more than half a century ago. The result shows a relation between the Jordan structures of two matrices AB and BA . The Jordan structures corresponding to the nonzero spectral points are the same. The situation at the zero eigenvalue is, however, more complicated. If $(n_j)_{j=0}^\infty$ and $(m_j)_{j=0}^\infty$

are decreasing sequences of sizes of Jordan blocks of AB and BA , respectively, corresponding to the zero eigenvalue, extended by an infinite number of zeros, then

$$|n_j - m_j| \leq 1, \quad j = 0, 1, \dots \quad (1.1)$$

Our second aim was to prove an analogue of the aforementioned for operators of the form $T^{[\natural]}T$ and $TT^{[\natural]}$ acting in an infinite dimensional Pontryagin space.

A (not complete) analysis of Jordan structures and canonical forms [7] of the pair of operators $T^{[\natural]}T$ and $TT^{[\natural]}$ in a finite dimensional space was done in [12]. Several special cases were treated there. In [16, Theorem 4.2], a complete description of the relations between T , $T^{[\natural]}$, $T^{[\natural]}T$ and $TT^{[\natural]}$ is given for the finite dimensional case. The approach taken there is to consider analogues of singular value decompositions of matrices in indefinite inner product spaces (see also [3]). Also the real case is treated there, as well as several cases with other symmetries. In [17] the relation to polar decomposition of this problem was discussed: in the finite dimensional case it turns out that an operator T admits polar decomposition if and only if $T^{[\natural]}T$ and $TT^{[\natural]}$ have the same canonical form with respect to the given indefinite inner product. Polar decomposition in Pontryagin spaces was discussed in [17] (for normal operators in the indefinite inner product) and [15]. The latter paper also contains some results on operators of the form $T^{[\natural]}T$, which we shall use later on.

In the present paper we focus attention mostly on those aspects that really belong to the infinite dimensional situation, opposed to the finite dimensional case. The outcome of this paper can be summarized as follows. In Section 4 we prove that for nonzero $\lambda \in \mathbb{R}$ the properties of λ as a spectral point of $T^{[\natural]}T$ and $TT^{[\natural]}$ are strongly related. It is also shown that the nonreal spectra of the operators coincide. The spectral point zero is considered in Sections 4, 6. It appears that the four possibilities mentioned above can occur in almost any combination for $T^{[\natural]}T$ and $TT^{[\natural]}$, as can be seen from the following table. The table is obviously

$TT^{[\natural]} \setminus T^{[\natural]}T$	$0 \in \rho$	$0 \in \sigma_+ \cup \sigma_-$	0 is reg. crit.	0 is sing. crit.
$0 \in \rho$	possible	possible	possible Ex. 6.1	impossible Prop. 4.4
$0 \in \sigma_+ \cup \sigma_-$		possible	possible Ex. 6.2	possible Ex. 6.3
0 is reg. crit.			possible	possible Ex. 6.4
0 is sing. crit.				possible [15, Ex. 3.8], [11]

TABLE 1. Zero as a spectral point

symmetric with respect to the diagonal and the part under the diagonal was left empty for the sake of clarity. The fact that zero cannot be a resolvent point of

one of the operators and at the same time a singular critical point of the other operator is proved in Section 4. The examples showing that all other cases are possible even in the class of bounded operators in a separable Π_1 -space are given in Section 6.

The behavior of infinity as a spectral point (with the usual understanding of the four notions of resolvent point, critical point etc.) is also rather peculiar, and is investigated in Section 5. The results are presented in the next table.

$TT^{[\natural]} \setminus T^{[\natural]}T$	$\infty \in \rho$	$\infty \in \sigma_+ \cup \sigma_-$	∞ is reg. crit.	∞ is sing. crit.
$\infty \in \rho$	possible	impossible	impossible	impossible
$\infty \in \sigma_+ \cup \sigma_-$		possible	possible Ex. 5.1	possible Ex. 5.1
∞ is reg. crit.			impossible Prop. 5.1	impossible Prop. 5.1
∞ is sing. crit.				impossible Prop. 5.1

TABLE 2. Infinity as a spectral point

In the last section we work in a Pontryagin space. This assumption assures us that at each eigenvalue there is only a finite number of Jordan chains longer than one for each of the operators. Hence, we are able to compare the Jordan structures of the operators at each eigenvalue. A reduction argument allows us to apply the theorem of Flanders and to obtain a similar result (Theorem 7.2).

2. Preliminaries

In the whole paper $(\mathcal{K}, [\cdot, \cdot])$ stands for a Krein space (in Sections 6 and 7 it will be a Pontryagin space). At this point we fix one of the complete norms on \mathcal{K} such that the inner product is continuous, and denote it by $\|\cdot\|$. Note that all such norms are equivalent (see [1, 14]), and none of the arguments below depend on a choice of an equivalent norm. If A is an operator in \mathcal{K} then by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ we understand the domain, the kernel and the range of A , respectively. The sum and product of unbounded operators is understood in a standard way, see e.g. [5, 19]. We write $\mathbf{B}(\mathcal{K})$ for the space of all bounded operators with domain equal \mathcal{K} . $A^{[\natural]}$ denotes the adjoint of a densely defined operator A in a Krein space. As usual, $\sigma(A)$ ($\rho(A)$) stand for the spectrum (the resolvent set) of a closed, densely defined operator A in \mathcal{K} .

Let A be a selfadjoint operator in \mathcal{K} . We call A *definitizable* if $\rho(A) \neq \emptyset$ and there exists a (real or complex) polynomial p such that $[p(A)f, f] \geq 0$ for $f \in \mathcal{D}(p(A))$. Any polynomial p satisfying the last inequality is called a *definitizing polynomial* for A . Note that even the degree of a definitizing polynomial is usually not unique (see [14, 10] for all results mentioned in this paragraph). We define the

set of *critical points* of a definitizable operator A as $c(A) := c_0(A) \cap \sigma(A) \cap \mathbb{R}$, where

$$c_0(A) := \bigcap_{p \text{ definitizing for } A} p^{-1}(0).$$

It is well known that $c(A) = c_0(A) \cap \mathbb{R}$. The Jordan chains corresponding to an eigenvalue λ of a definitizable operator are not longer than $k(\lambda) + 1$ ($k(\lambda)$ for nonreal λ), where $k(\lambda)$ is the multiplicity of λ as a zero of (any) definitizing polynomial p . This is the same as to say that *the algebraic root space*

$$\mathcal{S}_\lambda(A) := \{f \in \mathcal{D}(A) : \exists n \in \mathbb{N} \setminus \{0\} : (A - \lambda)^n f = 0\}$$

equals $\mathcal{N}((A - \lambda)^{k(\lambda)+1})$.

By $\mathfrak{R}(A)$ we understand the semiring generated by finite intervals and their complements with endpoints not in $c(A)$. On this semiring we define the spectral mapping E of the operator A , see [14] for the definition and properties. The definition which we use involves contour integrals (as in [14]), although similar results could be obtained also with the usage of functional calculus from [10]. By $\sigma_+(A)$ ($\sigma_-(A)$) we denote the set of all $\lambda \in \sigma(A) \cap \mathbb{R}$ for which there exists an interval $\tau \in \mathfrak{R}(A)$, $\lambda \in \tau$, such that $\mathcal{R}(E(\tau))$ is a positive (negative) subspace of \mathcal{K} . Critical points are those points λ of the real spectrum for which the space $\mathcal{R}(E(\tau))$ is indefinite for any neighborhood τ of λ . We call a critical point λ *regular* if the limits $\lim_{x \uparrow \lambda} E([\lambda_0, x])$ and $\lim_{x \downarrow \lambda} E([x, \lambda_1])$ exist in the strong operator topology for any (some) not critical $\lambda_0 \leq \lambda$, $\lambda_1 \geq \lambda$. This is equivalent to saying that for every neighborhood τ of λ such that $\tau \cap c(A) = \{\lambda\}$ and the spectral function E is bounded on subsets of τ [14, Theorem 5.7], or, again equivalently, that there exists a neighborhood τ with $\tau \cap c(A) = \{\lambda\}$ such that the spectral function E is bounded on subsets of τ . We call a critical point *singular*, if it is not regular.

An isolated point of the real spectrum is either in the definite part $\sigma_+(A) \cup \sigma_-(A)$ of the spectrum or is a regular critical point. In each case the spectral projection $E(\{\lambda\})$ (understood as a limit) equals the Riesz's projection [5] onto the algebraic root subspace corresponding to the eigenvalue λ .

3. Operators of the form $T^{\sharp}T$ in a Krein space

Let us say some words on the operators of the form $T^{\sharp}T$ in the unbounded, Krein space case. Such an operator is naturally symmetric in the sense that $[T^{\sharp}Tf, f] \in \mathbb{R}$ for $f \in \mathcal{D}(T^{\sharp}T)$, but it does not have to be even densely defined. Consider the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and introduce on the space $\mathcal{K} = \mathcal{H} \times \mathcal{H}$ the Krein space inner product: $[(f, g), (h, k)] := \langle f, k \rangle + \langle g, h \rangle$, $f, g, h, k \in \mathcal{H}$. Now let us take $T = A \oplus B$, where A and B are both closed, densely defined operators in \mathcal{H} (i.e. $T(f, g) := (Af, Bg)$, $(f, g) \in \mathcal{D}(T) = \mathcal{D}(A) \times \mathcal{D}(B)$). It is clear that

$$T^{\sharp}T = (B^*A) \oplus (A^*B), \quad TT^{\sharp} = (AB^*) \oplus (BA^*).$$

If we choose as A any unbounded selfadjoint (in \mathcal{H}) operator and set $B = \langle \cdot, e \rangle f$ with $f \in \mathcal{D}(A)$, $e \notin \mathcal{D}(A)$ then $T^{[\natural]}T$ is densely defined while $TT^{[\natural]}$ is not.

Consider now the following conditions, for a densely defined and closed operator T in a Krein space \mathcal{K} :

- (t1) $T^{[\natural]}T$ and $TT^{[\natural]}$ are (densely defined and) selfadjoint operators in \mathcal{K} .
- (t2) $T^{[\natural]}T$ and $TT^{[\natural]}$ have nonempty resolvent sets.
- (t3) $T^{[\natural]}T$ is definitizable.

Theorem 3.1. *Under the conditions (t1) and (t2) the operator $T^{[\natural]}T$ is definitizable if and only if $TT^{[\natural]}$ is definitizable. Moreover, if $p(t)$ is a definitizing polynomial for $T^{[\natural]}T$ then $tp(t)$ is a definitizing polynomial for $TT^{[\natural]}$. Consequently,*

$$c_0(T^{[\natural]}T) \cup \{0\} = c_0(TT^{[\natural]}) \cup \{0\}. \quad (3.1)$$

Proof. For $f \in \mathcal{D}((TT^{[\natural]})p(TT^{[\natural]}))$ we have

$$[(TT^{[\natural]})p(TT^{[\natural]})f, f] = [T^{[\natural]}p(TT^{[\natural]})f, T^{[\natural]}f] = [p(T^{[\natural]}T)T^{[\natural]}f, T^{[\natural]}f] \geq 0.$$

This proves the first two sentences of the theorem. In consequence, $c_0(TT^{[\natural]}) \cup \{0\} \subseteq c_0(T^{[\natural]}T) \cup \{0\}$. We obtain the converse inclusion by interchanging the roles of T and $T^{[\natural]}$. \square

From now on we assume that T satisfies (t1)–(t3). The last example and theorem show our reasons for this assumption. Another motivation is that in a Pontryagin space (t1)–(t3) are always satisfied for a densely defined, closed T . This comes from the fact, that by [13] the second power of a selfadjoint operator is selfadjoint and we can apply Nelson's trick [18, top of page 143]. To be precise, let T be a densely defined operator in a Π_κ -space $(\mathcal{K}, [\cdot, \cdot])$. Consider the space $\mathcal{K} \times \mathcal{K}$ with the $\Pi_{2\kappa}$ -inner product $[(f, g), (h, k)] = [f, h] + [g, k]$ ($f, g, h, k \in \mathcal{K}$) and the operator

$$Q(f, g) = (T^{[\natural]}g, Tf), \quad (f, g) \in \mathcal{D}(Q) = \mathcal{D}(T) \times \mathcal{D}(T^{[\natural]}).$$

Since Q is selfadjoint in $\mathcal{K} \times \mathcal{K}$ so is Q^2 , by the result of Langer [13]. Hence, both $T^{[\natural]}T$ and $TT^{[\natural]}$ are selfadjoint (and thus definitizable).

4. Definitizable operators in Krein spaces.

Positive results on types of spectral point

Now let (t1)–(t3) hold and let E and E_* denote the spectral function of $T^{[\natural]}T$ and $TT^{[\natural]}$ respectively. By \mathfrak{R}_0 we denote the semiring generated by finite intervals and their complements with endpoints not in $c(T^{[\natural]}T) \cup \{0\}$. We put \mathfrak{R}_0^0 for the family of all its bounded elements.

Theorem 4.1. *The following inclusions hold*

$$E_*(\tau)T \subseteq TE(\tau), \quad E(\tau)T^{[\mathfrak{h}]} \subseteq T^{[\mathfrak{h}]}E_*(\tau), \quad \tau \in \mathfrak{R}_0. \quad (4.1)$$

Moreover, for $\tau \in \mathfrak{R}_0^0$ we have

$$\overline{E_*(\tau)T} = TE(\tau) \in \mathbf{B}(\mathcal{K}), \quad \overline{E(\tau)T^{[\mathfrak{h}]}} = T^{[\mathfrak{h}]}E_*(\tau) \in \mathbf{B}(\mathcal{K}). \quad (4.2)$$

Proof. We prove only the first inclusion in (4.1), the proof of the second one is similar. First let $\tau \in \mathfrak{R}_0^0$. By definition,

$$E_*(\tau)h = -\frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{C_{\tau_\varepsilon}^\delta} (TT^{[\mathfrak{h}]} - z)^{-1} h \, dz, \quad h \in \mathcal{K},$$

where the contour C_σ^δ and the set τ_ε are defined as in [14] (see also [19]). The integral is understood in the strong sense as a limit of Riemann sums

$$\lim_{\Delta P \rightarrow 0} \sum_{j=1}^{N(P)} (t_j - t_{j-1}) \phi'_{\delta\varepsilon}(s_j) (TT^{[\mathfrak{h}]} - \phi_{\delta\varepsilon}(s_j))^{-1} h,$$

where $\phi_{\delta\varepsilon} : [0, 1] \rightarrow \mathbb{C}$ is a parametrization of $C_{\tau_\varepsilon}^\delta$, $P = (t_0, \dots, t_{N(P)})$ is a partition of $[0, 1]$, ΔP is the mesh of the partition and $s_j \in [t_{j-1}, t_j]$. Now for $f \in \mathcal{D}(T)$ we have

$$\int_{C_{\tau_\varepsilon}^\delta} (TT^{[\mathfrak{h}]} - z)^{-1} T f \, dz = \lim_{\Delta P \rightarrow 0} T \sum_{j=1}^{N(P)} (t_j - t_{j-1}) \phi'_{\delta\varepsilon}(s_j) (T^{[\mathfrak{h}]}T - \phi_{\delta\varepsilon}(s_j))^{-1} f,$$

because $(TT^{[\mathfrak{h}]} - z)^{-1} T f = T(T^{[\mathfrak{h}]}T - z)^{-1} f$ if only both resolvent operators exist in $\mathbf{B}(\mathcal{K})$. Since T is a closed operator, the element

$$\int_{C_{\tau_\varepsilon}^\delta} (T^{[\mathfrak{h}]}T - z)^{-1} f \, dz = \lim_{\Delta P \rightarrow 0} \sum_{j=1}^{N(P)} (t_j - t_{j-1}) \phi'_{\delta\varepsilon}(s_j) (T^{[\mathfrak{h}]}T - \phi_{\delta\varepsilon}(s_j))^{-1} f$$

is in the domain of T and

$$\begin{aligned} T \int_{C_{\tau_\varepsilon}^\delta} (T^{[\mathfrak{h}]}T - z)^{-1} f \, dz &= \lim_{\Delta P \rightarrow 0} T \sum_{j=1}^{N(P)} (t_j - t_{j-1}) \phi'_{\delta\varepsilon}(s_j) (T^{[\mathfrak{h}]}T - \phi_{\delta\varepsilon}(s_j))^{-1} f \\ &= \int_{C_{\tau_\varepsilon}^\delta} (TT^{[\mathfrak{h}]} - z)^{-1} T f \, dz. \end{aligned}$$

In the same way we can interchange T with the other two limits in the definition of $E_*(\tau)$ and get that $E_*(\tau)Tf = TE(\tau)f$ for $f \in \mathcal{D}(T) = \mathcal{D}(E_*(\tau)T)$. This proves the first of the inclusions (4.1) in the case $\tau \in \mathfrak{R}_0^0$. Note that in this case we also have (see [14, Theorem 3.1(6)])

$$\mathcal{R}(E(\tau)) \subseteq \mathcal{D}(T^{[\mathfrak{h}]}T) \subseteq \mathcal{D}(T). \quad (4.3)$$

Since T is a closed operator we get also the first part of (4.2).

By definition we have $E(\mathbb{R} \setminus \tau) = I - E(\tau)$ ($\tau \in \mathfrak{R}_0^0$). Note that $E(\mathbb{R} \setminus \tau)(\mathcal{D}(T)) \subseteq \mathcal{D}(T)$ and so $\mathcal{D}(E_*(\mathbb{R} \setminus \tau)T) = \mathcal{D}(T) \subseteq \mathcal{D}(TE(\mathbb{R} \setminus \tau))$. Moreover, for $f \in \mathcal{D}(T)$ we have

$$E_*(\mathbb{R} \setminus \tau)Tf = (I - E_*(\tau))Tf = Tf - E_*(\tau)Tf = Tf - TE(\tau)f = TE(\mathbb{R} \setminus \tau)f.$$

□

Theorem 4.2. *Let $\lambda \in \mathbb{R} \setminus \{0\}$. Then*

- (i) $\lambda \in \rho(T^{[\mathfrak{h}]}T)$ if and only if $\lambda \in \rho(TT^{[\mathfrak{h}]})$,
- (ii) λ is a critical point of $T^{[\mathfrak{h}]}T$ if and only if it is a critical point of $TT^{[\mathfrak{h}]}$,
- (iii) λ is a regular critical point of $T^{[\mathfrak{h}]}T$ if and only if it is a regular critical point of $TT^{[\mathfrak{h}]}$.

Proof. (i) Let $\lambda \in \tau \subseteq \rho(T^{[\mathfrak{h}]}T)$ for some open $\tau \in \mathfrak{R}_0^0$. Since $E(\tau) = 0$, we have $E(\tau)T^{[\mathfrak{h}]}f = 0$ for $f \in \mathcal{D}(T^{[\mathfrak{h}]})$. From (4.2) we get $T^{[\mathfrak{h}]}E_*(\tau) = 0$ and consequently $TT^{[\mathfrak{h}]}|_{\mathcal{R}(E_*(\tau))} = 0$. But on the other hand $0 \in \rho(TT^{[\mathfrak{h}]}|_{\mathcal{R}(E_*(\tau))})$, because $\sigma(TT^{[\mathfrak{h}]}|_{\mathcal{R}(E_*(\tau))}) \subseteq \bar{\tau}$, [14, Theorem 3.1(7)]. Therefore, $E_*(\tau) = 0$ and $\lambda \in \rho(TT^{[\mathfrak{h}]})$.

Point (ii) follows from Theorem 3.1 and the fact that $c(A) = c_0(A) \cap \mathbb{R}$ for definitizable A . Let us turn to the proof of (iii). Suppose that λ is a regular critical point of $T^{[\mathfrak{h}]}T$ and let us take a bounded closed neighborhood τ of λ such that $\tau \cap (c(T^{[\mathfrak{h}]}T) \cup \{0\}) = \emptyset$. Since λ is a regular critical point of $T^{[\mathfrak{h}]}T$ the spectral function E is bounded on subsets of τ , i.e. there exists a constant $c \geq 0$ such that

$$\|E(\sigma)\| \leq c, \quad \sigma \subseteq \tau, \quad \sigma \in \mathfrak{R}_0. \quad (4.4)$$

To prove that λ is a regular critical point for $TT^{[\mathfrak{h}]}$ it is enough to show that E_* is bounded on the subsets of τ , [14]. First set

$$\mathcal{R}_1 := \mathcal{R}(E(\tau)), \quad \mathcal{R}_2 := \mathcal{R}(E_*(\tau)),$$

and note that [14, p. 30(6)]

$$T^{[\mathfrak{h}]}T|_{\mathcal{R}_1} \in \mathbf{B}(\mathcal{R}_1).$$

Since $0 \notin \tau$ we have that $0 \in \rho(T^{[\mathfrak{h}]}T|_{\mathcal{R}_1})$ [14, p. 30(7)] which means that $T^{[\mathfrak{h}]}T$ is a bijection from \mathcal{R}_1 onto itself. Similarly, $TT^{[\mathfrak{h}]}$ is a bijection from \mathcal{R}_2 onto itself.

On the other hand Theorem 4.1 shows that

$$T^{[\mathfrak{h}]}(\mathcal{R}_2) \subseteq \mathcal{R}_1, \quad T(\mathcal{R}_1) \subseteq \mathcal{R}_2. \quad (4.5)$$

Hence $\mathcal{R}_2 = T(T^{[\mathfrak{h}]}(\mathcal{R}_2)) \subseteq T(\mathcal{R}_1) \subseteq \mathcal{R}_2$ and consequently $T(\mathcal{R}_1) = \mathcal{R}_2$. Similarly $T^{[\mathfrak{h}]}(\mathcal{R}_2) = \mathcal{R}_1$. Therefore, T is a bijection from \mathcal{R}_1 onto \mathcal{R}_2 (and $T^{[\mathfrak{h}]}$ is a bijection from \mathcal{R}_2 onto \mathcal{R}_1). Now for $\sigma \subseteq \tau$ such that $\lambda \in \sigma \in \mathfrak{R}_0$ we get, since $\mathcal{R}(E_*(\sigma)) \subseteq \mathcal{R}_2$,

$$\begin{aligned} \|E_*(\sigma)\| &= \|E_*(\sigma)E_*(\tau)\| = \|(T|_{\mathcal{R}_2})^{-1}T|_{\mathcal{R}_2} E_*(\sigma)E_*(\tau)\| \\ &\leq \|(T|_{\mathcal{R}_2})^{-1}\| \|E(\sigma)T|_{\mathcal{R}_2} E_*(\tau)\| \\ &\leq \|(T|_{\mathcal{R}_2})^{-1}\| c \|T|_{\mathcal{R}_2} E_*(\tau)\|. \end{aligned}$$

□

To have a complete picture let us deal in this moment with the nonreal spectra.

Proposition 4.3. *Let $\tau = \{\lambda\}$ with $\lambda \notin \mathbb{R}$. Then the formulas in Theorem 4.1 hold as well, with the interpretation that $E(\{\lambda\})$ and $E_*(\{\lambda\})$ are the Riesz's projections onto the algebraic root spaces $S_\lambda(T^{[\natural]}T)$ and $S_\lambda(TT^{[\natural]})$, respectively. Consequently, the nonreal spectra of $T^{[\natural]}T$ and $TT^{[\natural]}$ coincide.*

Proof. The proofs of the mentioned formulas are the same as the proof of Theorem 4.1, only the limits in δ and ε are not necessary. Now let $\lambda \in (\rho(T^{[\natural]}T) \cap \sigma(TT^{[\natural]})) \setminus \mathbb{R}$. Since each nonreal point of spectrum is necessarily a common zero of all definitizing polynomials [14, p. 28], λ must be in $c_0(TT^{[\natural]})$. By the first part of this proposition we have $T^{[\natural]}E_*(\{\lambda\}) = 0$, and consequently $TT^{[\natural]}|_{\mathcal{R}(E_*(\{\lambda\}))} = 0$. Since $\lambda \neq 0$, we have $\mathcal{R}(E(\{\lambda\})) = \{0\}$. \square

The only result about the zero eigenvalue we can prove is the following.

Proposition 4.4. *If 0 is in the resolvent of $T^{[\natural]}T$ then it is not a singular critical point of $TT^{[\natural]}$.*

Proof. Since the resolvent set is open and the nonzero spectrum of $T^{[\natural]}T$ is equal to the nonzero spectrum of $TT^{[\natural]}$, zero is an isolated point of spectrum of $TT^{[\natural]}$ and thus it can not be a singular critical point. \square

5. Analysis of infinity as a spectral point

Let A be a definitizable operator. We write $\infty \in \rho(A)$ if and only if A is bounded (equivalently, if $\sigma(A)$ is bounded, see [14, 10]). We say that infinity is in the positive (negative) spectrum if there exists a real neighborhood of infinity τ such that $E(\tau)$ is positive (negative). We call infinity a *critical point* of a definitizable operator A if in each real neighborhood of infinity there exist points from both $\sigma_+(S)$ and $\sigma_-(S)$. If infinity is a critical point we call it *regular* if the limits $\lim_{x \uparrow +\infty} E([\lambda, x])$ and $\lim_{x \downarrow -\infty} E([x, \lambda])$ exist in the strong operator topology for any (some) not critical $\lambda \in \mathbb{R}$, otherwise we call it *singular*.

Let us assume now (t1)–(t3) and look at Table 2. The reader can surely find examples for $\infty \in \sigma_+(T^2)$ and $\infty \in \rho(T^2)$, with T selfadjoint in a Hilbert space. By virtue of the results of the previous section (Theorem 4.2), we get $\infty \in \rho(T^{[\natural]}T) \Leftrightarrow \infty \in \rho(TT^{[\natural]})$, which completes the first row of the table. A direct consequence of Theorem 3.1 and Theorem 4.2 (and [14, Theorem 3.1(4)]) is the following proposition; we use the notation $\mathbb{R}_\pm := \{x \in \mathbb{R} : \pm x > 0\}$.

Proposition 5.1. *We have*

$$\sigma_\pm(T^{[\natural]}T) \cap \mathbb{R}_+ = \sigma_\pm(TT^{[\natural]}) \cap \mathbb{R}_+, \quad \sigma_\pm(T^{[\natural]}T) \cap \mathbb{R}_- = \sigma_\mp(TT^{[\natural]}) \cap \mathbb{R}_-. \quad (5.1)$$

Consequently, infinity can be a critical point of at most one of the operators $T^{[\natural]}T$ and $TT^{[\natural]}$.

What remains to complete Table 2, is to show that infinity indeed can be a singular or regular critical point of an operator of the form $T^{[n]}T$.

Example 5.1. Let us take a positive operator A with a singular (regular) critical point at infinity and $0 \in \rho(A)$ in a separable Krein space \mathcal{K} with an infinite dimensional, uniformly positive [1] subspace \mathcal{K}_+ (e.g. take the one from [4] for the singular critical point, the regular case is left to reader as a simple exercise). We will show now that $A = T^{[n]}T$ for some closed, densely defined T (cf. [1, Theorem VII.3.1] for the bounded case).

Since A is positive and invertible, the space $(\mathcal{D}(A), [A\cdot, \cdot])$ is a unitary space. Indeed, if $f \in \mathcal{D}(A)$ is such that $[Af, f] = 0$ then, by the Schwartz inequality, $[Af, g] = 0$ for $g \in \mathcal{D}(A)$. Consequently, $Af = 0$ and so $f = 0$. Note that the graph $\Gamma(A) = \{(f, Af) : f \in \mathcal{D}(A)\}$, with the topology inherited from $\mathcal{K} \times \mathcal{K}$, is separable, as a closed subspace of a separable Hilbert (Krein) space. The linear mapping $\Gamma(A) \ni (f, Af) \mapsto f \in \mathcal{D}(A)$ is onto and continuous with respect to the $[A\cdot, \cdot]$ -inner product topology on $\mathcal{D}(A)$, since $[Af, f] \leq c \|Af\| \|f\| \leq c(\|Af\|^2 + \|f\|^2)$ ($f \in \mathcal{D}(A)$) for some $c \geq 0$. Hence, $(\mathcal{D}(A), [A\cdot, \cdot])$ is separable as well. Let us take $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ as the completion of the unitary space $(\mathcal{D}(A), [A\cdot, \cdot])$ to a Hilbert space. Since $\mathcal{D}(A)$ is dense in \mathcal{H} , \mathcal{H} is a separable Hilbert space. Therefore, there exists an isometric mapping U from the separable Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ into the separable Hilbert space $(\mathcal{K}_+, [\cdot, \cdot])$. We define the operator \tilde{T} in \mathcal{K} as $\tilde{T}f = Uf$ ($f \in \mathcal{D}(A)$) and we denote by T its closure. Observe that

$$[Tf, Tg] = [Af, g], \quad f, g \in \mathcal{D}(A).$$

If we fix $f \in \mathcal{D}(A)$ in the above, we get $Tf \in \mathcal{D}(T^{[n]})$ and $T^{[n]}Tf = Af$. This shows that $T^{[n]}T \supseteq A$. The operator on the left hand side is symmetric and the one on the right hand side is selfadjoint, hence $T^{[n]}T = A$.

6. Zero as a spectral point. Counterexamples in Π_1 -spaces

The last two sections concern Pontryagin spaces. In this section we discuss the results indicated in Table 1, insofar as they have not already been proved in Section 4. More information on the dimensions of the algebraic root spaces will be given in the next section.

Recall that for T closed and densely defined in a Π_κ -space the assumptions (t1)–(t3) are fulfilled.

Let us now start completing Table 1. Proposition 4.4 has already been proved. Observe that there are obvious examples (in the class of bounded operators in a Hilbert space) for $0 \in \rho(T^{[n]}T) \cap \rho(TT^{[n]})$, $0 \in \sigma_+(T^{[n]}T) \cap \sigma_+(TT^{[n]})$, $0 \in \sigma_+(T^{[n]}T) \cap \rho(TT^{[n]})$. If T is a zero operator in a 2-dimensional Π_1 -space then zero is a regular critical point for $T^{[n]}T = TT^{[n]}$. Now let us turn to more complicated examples.

Example 6.1. Zero is in the resolvent of $T^{[k]}T$ and is a regular critical point of $TT^{[k]}$.

Let us consider the Hilbert space $\ell^2(\mathbb{Z} \setminus \{0\})$ and the fundamental symmetry $J \in \mathbf{B}(\ell^2(\mathbb{Z} \setminus \{0\}))$, which is uniquely determined below by its action on the canonical basis $(e_j)_{j \in \mathbb{Z} \setminus \{0\}}$,

$$J(e_j) = \begin{cases} e_j & : |j| > 1 \\ e_{-1} & : j = 1 \\ e_1 & : j = -1 \end{cases} \quad j \in \mathbb{Z} \setminus \{0\}.$$

It is clear, that $\ell^2(\mathbb{Z} \setminus \{0\})$ with the inner product $[\cdot, \cdot] = \langle J\cdot, \cdot \rangle$ is a Π_1 space. We define the operator $T \in \mathbf{B}(\ell^2(\mathbb{Z} \setminus \{0\}))$ by

$$T(e_j) = \begin{cases} e_{j+1} & : j > 0 \\ e_{j-1} & : j < 0 \end{cases} \quad j \in \mathbb{Z} \setminus \{0\}.$$

It is easy to verify that $T^{[k]} = JT^*J$ acts in the following way:

$$T^{[k]}(e_j) = \begin{cases} 0 & : j = 1, -1 \\ e_{-1} & : j = 2 \\ e_1 & : j = -2 \\ e_{j-1} & : j > 2 \\ e_{j+1} & : j < -2 \end{cases} \quad j \in \mathbb{Z} \setminus \{0\}.$$

Hence

$$T^{[k]}T(e_j) = \begin{cases} e_{-j} & : |j| = -1, 1 \\ e_j & : |j| > 1 \end{cases} \quad j \in \mathbb{Z} \setminus \{0\}$$

and

$$TT^{[k]}(e_j) = \begin{cases} 0 & : j = 1, -1 \\ e_{-j} & : j = 2, -2 \\ e_j & : |j| > 2 \end{cases} \quad j \in \mathbb{Z} \setminus \{0\}.$$

Observe that $T^{[k]}T$ can be viewed as a block operator matrix acting on a Hilbert space $\ell^2(\mathbb{Z} \setminus \{0\})$ as follows:

$$T^{[k]}T = I \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I.$$

From this it is apparent that $\sigma(T^{[k]}T) = \{-1, 1\}$ and likewise, $\sigma(TT^{[k]}) = \{0, -1, 1\}$. Moreover, the algebraic root subspace corresponding to the zero eigenvalue of $TT^{[k]}$ satisfies

$$\mathcal{S}_0(T^{[k]}T) = \mathcal{N}(T^{[k]}T) = \text{lin}\{e_1, e_{-1}\}.$$

Hence, it is a nondegenerate indefinite subspace and zero is a regular critical point of $TT^{[k]}$.

In the example above all Jordan chains for the zero eigenvalue for $TT^{[k]}$ have length one. In fact, this is the maximal length if zero is in the resolvent of $T^{[k]}T$, by Theorem 7.2 in the next section. The next example illustrates the situation, when zero is in $\sigma_+(TT^{[k]})$ and is a regular critical point of $T^{[k]}T$. Moreover, there

is a Jordan chain of length two corresponding to the zero eigenvalue of $TT^{[\natural]}$ and the operator T acts on a Π_1 -space.

Example 6.2. Let us consider the space ℓ^2 and the Π_1 -inner product on ℓ^2 given by the fundamental symmetry $J \in \mathbf{B}(\ell^2)$, which is defined on the canonical basis $(e_j)_{j=0}^\infty$ as

$$J(e_j) = \begin{cases} e_2 & : j = 0 \\ e_0 & : j = 2 \\ e_j & : j \neq 0, 2 \end{cases} \quad j \in \mathbb{N}.$$

We define the operator T by $T(e_j) = e_{j+1}$ ($j \in \mathbb{N}$). It is easy to compute that $T^{[\natural]} = JT^*J$ satisfies

$$T^{[\natural]}(e_j) = \begin{cases} e_{j+1} & : j = 0, 1 \\ 0 & : j = 2 \\ e_0 & : j = 3 \\ e_{j-1} & : j > 3 \end{cases} \quad j \in \mathbb{N}$$

and hence

$$T^{[\natural]}T(e_j) = \begin{cases} e_2 & : j = 0 \\ 0 & : j = 1 \\ e_0 & : j = 2 \\ e_j & : j > 2 \end{cases} \quad j \in \mathbb{N}$$

and

$$TT^{[\natural]}(e_j) = \begin{cases} e_{j+2} & : j = 0, 1 \\ 0 & : j = 2 \\ e_1 & : j = 3 \\ e_j & : j > 3. \end{cases} \quad j \in \mathbb{N}.$$

It is apparent that $\sigma(T^{[\natural]}T) = \sigma(TT^{[\natural]}) = \{-1, 0, 1\}$. However, 0 is in the positive part of spectrum of $T^{[\natural]}T$, since

$$\mathcal{S}_0(T^{[\natural]}T) = \mathcal{N}(T^{[\natural]}T) = \text{lin}\{e_1\},$$

which is a positive space. On the other hand, (e_0, e_2) is a Jordan chain for the eigenvalue 0 of $TT^{[\natural]}$ and the space

$$\mathcal{S}_0(TT^{[\natural]}) = \mathcal{N}((TT^{[\natural]})^2) = \text{lin}\{e_0, e_2\}$$

is indefinite and nondegenerate. Hence, 0 is a regular critical point for $TT^{[\natural]}$.

The example [15, Example 3.8] (see also [11]) shows a bounded, selfadjoint operator T in a Π_1 -space such that zero is a singular critical point of T^2 . This is the lower-right corner of Table 1. A modification of that example, which is shown below, leads to a situation where zero is in the positive part of spectrum of $T^{[\natural]}T$ and is a singular critical point of $TT^{[\natural]}$.

Example 6.3. Let \mathcal{K} be the Hilbert space $L^2[0, 1] \oplus \mathbb{C}^2 \oplus \ell^2$ with the natural scalar product $\langle \cdot, \cdot \rangle$. We define the fundamental symmetry $J(f, x, y, l) = (f, y, x, l)$ for all $f \in L^2[0, 1]$, $x, y \in \mathbb{C}$, $l \in \ell^2$. Obviously, $(\mathcal{K}, \langle J \cdot, \cdot \rangle)$ is a Π_1 -space. Consider the operator

$$T := \begin{pmatrix} M_{\sqrt{t}} & 0 & \pi(\mathbf{1}) & 0 \\ \langle \cdot, \mathbf{1} \rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \pi(e_1) & 0 & S \end{pmatrix}$$

where $M_\phi \in \mathbf{B}(L^2[0, 1])$ denotes the multiplication operator by a bounded function ϕ , S is the shift operator in ℓ^2 ($Se_j = e_{j+1}$, $j \in \mathbb{N}$), $\pi(g)$ (where g is an element of some Hilbert space) maps $x \in \mathbb{C}$ to xg and $\mathbf{1} \in L^2[0, 1]$ is a function constantly equal one. It is not hard to compute that

$$\begin{aligned} T^{[i]} &= J \begin{pmatrix} M_{\sqrt{t}} & \pi(\mathbf{1}) & 0 & 0 \\ 0 & 0 & 0 & \langle \cdot, e_1 \rangle \\ \langle \cdot, \mathbf{1} \rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & S^* \end{pmatrix} J \\ &= \begin{pmatrix} M_{\sqrt{t}} & 0 & \pi(\mathbf{1}) & 0 \\ \langle \cdot, \mathbf{1} \rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & \langle \cdot, e_1 \rangle \\ 0 & 0 & 0 & S^* \end{pmatrix}. \end{aligned}$$

Next we compute

$$T^{[i]}T = \begin{pmatrix} M_t & 0 & \pi(\sqrt{t}) & 0 \\ \langle \cdot, \sqrt{t} \rangle & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{\ell^2} \end{pmatrix}.$$

(The zero in position (3,4) is because $\langle Sl, e_1 \rangle = 0$ for all $l \in \ell^2$; the zero at (4,2) is because $S^*e_1 = 0$.) Let us note that 0 is not an eigenvalue of $T^{[i]}T$. Indeed, if $T^{[i]}T(f, x, y, l) = 0$ for some $(f, x, y, l) \in \mathcal{K}$ then in particular

$$\begin{aligned} tf(t) + y\sqrt{t} &= 0 \quad \text{a.e. in } t \text{ on } [0, 1], \\ x &= 0, \\ l &= 0. \end{aligned}$$

If $y \neq 0$ then the first equation does not have any solution in $f \in L^2[0, 1]$. Hence $y = 0$ and consequently $f = 0$. And so we proved that zero is not in the point spectrum of $T^{[i]}T$. Since this $T^{[i]}T$ is selfadjoint operator in a Pontryagin space, we know that zero is either in the positive spectrum or in the resolvent of $T^{[i]}T$. The latter option is not possible, since we will now prove that zero is a singular critical point of $TT^{[i]}$ and hence, by Proposition 4.4, it cannot be in $\rho(T^{[i]}T)$.

A simple computation shows that

$$TT^{[h]} = \begin{pmatrix} M_t & 0 & \pi(\sqrt{t}) & \langle \cdot, e_1 \rangle \mathbf{1} \\ \langle \cdot, \sqrt{t} \rangle & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \langle \cdot, \mathbf{1} \rangle e_1 & 0 & 0 & SS^* \end{pmatrix}.$$

An element (f, x, y, l) of the kernel of $TT^{[h]}$ satisfies

$$\begin{aligned} tf(t) + y\sqrt{t} + \langle l, e_1 \rangle &= 0 \quad \text{a.e. in } t \text{ on } [0, 1], \\ \langle f, \sqrt{t} \rangle + y &= 0, \\ \langle f, \mathbf{1} \rangle e_1 + SS^*l &= 0. \end{aligned} \tag{6.1}$$

The first equation shows that $y = 0$ and

$$\langle l, e_1 \rangle = 0, \tag{6.2}$$

otherwise it has no solutions in $f \in L^2([0, 1])$. Hence, $f = 0$. Therefore, the last equation gives $l = ce_1$ with some $c \in \mathbb{C}$. By (6.2), $c = 0$. Therefore, the space $\{(0, x, 0, 0) : x \in \mathbb{C}\}$ is the kernel of $TT^{[h]}$. Observe that it is also the algebraic root space. Indeed, if $TT^{[h]}(f, x, y, l) = (0, 1, 0, 0)$ then (6.1) holds and consequently $f = 0$, $y = 0$ and $l = 0$. Note that $\{(0, x, 0, 0) : x \in \mathbb{C}\}$ is a degenerate subspace. Resuming, zero is in $\sigma_+(T^{[h]}T)$ and it is a singular critical point of $TT^{[h]}$.

Example 6.4. Consider the space $\mathcal{K} = L^2[0, 1] \oplus \mathbb{C}^2$ with the fundamental symmetry $J(f, x, y) = (f, y, x)$ for all $f \in L^2[0, 1]$, $x, y \in \mathbb{C}$, which makes it a Π_1 -space. The operator

$$T := \begin{pmatrix} M_{\sqrt{t}} & 0 & 0 \\ \langle \cdot, \mathbf{1} \rangle & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has the property that zero is a regular critical point for $T^{[h]}T$ and singular critical point for $TT^{[h]}$. The details are similar to the ones in the previous example and therefore they are left to the reader.

7. Operators $T^{[h]}T$ and $TT^{[h]}$ in Pontryagin spaces. Comparing the Jordan chains

The reasoning in this section is independent on the type of spectral point λ . We use only linear algebra combined with the information below concerning the lengths of the Jordan chains. The difference between non-critical, singular and regular critical points lies in the fact that the spectral projection $E(\{\lambda\})$ may or may not exist, but this does not influence the Jordan structure at all.

If A is a selfadjoint operator in a Π_κ space then all the Jordan chains are not longer than $2\kappa + 1$, i.e. $\mathcal{S}_\lambda(A) = \mathcal{N}((A - \lambda)^{2\kappa+1})$, [1, 14, 9]. For each nonreal spectral

point λ the algebraic root subspace $\mathcal{S}_\lambda(A)$ is finite dimensional and $\mathcal{S}_\lambda(A) \dot{+} \mathcal{S}_{\bar{\lambda}}(A)$ is nondegenerate (hence, it is a Pontryagin space).

The notion of a Jordan chain corresponding to an eigenvalue of A makes sense as well. Namely, for each eigenvalue λ of A there exists a decomposition $\mathcal{S}_\lambda(A) = \mathcal{K}_0 \dot{+} \mathcal{K}_1$ (formally, we should write \mathcal{K}_i^λ , $i = 0, 1$), such that both spaces \mathcal{K}_0 , \mathcal{K}_1 are invariant for A , \mathcal{K}_0 is finite dimensional, and $A|_{\mathcal{K}_1}$ has no Jordan chains longer than one. The construction given in [9, Theorem 7.2] is not unique and for our purposes we need to proceed in a slightly different way, although the aforementioned result guarantees that our reasoning makes sense. In particular we can define the *Segre characteristic* $(n_j)_{j=0}^\infty$ for the operator A and a point $\lambda \in \mathbb{C}$. Namely, as n_0, \dots, n_k we set the lengths of Jordan chains of $A|_{\mathcal{K}_0}$ in decreasing order. If the space \mathcal{K}_0 is trivial, we set $k = -1$. We put $n_j = 1$ for $j = k + 1, \dots, k + \dim \mathcal{K}_1$ and $n_j = 0$ for $j > k + \dim \mathcal{K}_1$, if $\dim \mathcal{K}_1 < \infty$. Obviously, in the finite dimensional case this definition agrees with the standard one (decreasing sequence of sizes of Jordan blocks extended by an infinite number of zeros).

Note that the linear space $\mathcal{S}_\lambda(A)/\mathcal{N}(A - \lambda)$ is finite dimensional. Moreover, the operator

$$[A]_\lambda : \mathcal{S}_\lambda(A)/\mathcal{N}(A - \lambda) \ni [f]_\lambda \mapsto [Af]_\lambda \in \mathcal{S}_\lambda(A)/\mathcal{N}(A - \lambda)$$

is well defined. Later on we will omit the subscripts λ and we will not distinguish (in notation) between the λI operators in the original and quotient space.

Lemma 7.1. *Let A be a selfadjoint operator in a Pontryagin space, let λ be an eigenvalue of A . There exists a one-to-one correspondence between the Jordan chains longer than one of $A|_{\mathcal{K}_0}$ and the Jordan chains of $[A]$, in the following sense: if $\{f_i^{(j)} : i = 0, \dots, n_j, j = 0, \dots, k\}$ is a basis for \mathcal{K}_0 such that*

$$(A - \lambda)f_i^{(j)} = f_{i-1}^{(j)} \quad (i = 1, \dots, n_j), \quad (A - \lambda)f_0^{(j)} = 0, \quad j = 0, \dots, k, \quad (7.1)$$

then $\{[f_i^{(j)}] : i = 1, \dots, n_j, j = 0, \dots, k, n_j \geq 1\}$ is a basis for $\mathcal{S}_\lambda(A)/\mathcal{N}(A - \lambda)$ and

$$([A] - \lambda)[f_i^{(j)}] = [f_{i-1}^{(j)}], \quad i = 1, \dots, n_j, \quad j = 0, \dots, k, \quad (7.2)$$

$$([A] - \lambda)[f_1^{(j)}] = [f_0^{(j)}] = 0, \quad j = 0, \dots, k. \quad (7.3)$$

Consequently, if $(n_j)_{j=0}^\infty$ is the Segre characteristic for A , then

$$\tilde{n}_j = \max\{n_j - 1, 0\}, \quad j = 0, 1, \dots, \quad (7.4)$$

is the Segre characteristic for $[A]$.

Proof. It is apparent, that (7.1) implies (7.2) and (7.3). For simplicity set

$$J := \{(i, j) : i = 1, \dots, n_j, j = 0, \dots, l, n_j \geq 1\}.$$

Now we prove that the vectors $\{[f_i^{(j)}] : (i, j) \in J\}$ are linearly independent. Indeed, if

$$\sum_{(i,j) \in J} \alpha_{ij} [f_i^{(j)}] = 0$$

for some complex numbers α_{ij} with $(i, j) \in J$ then

$$\sum_{(i,j) \in J} \alpha_{ij} f_i^{(j)} \in \mathcal{N}(A - \lambda).$$

But since $\{f_i^{(j)} : i = 0, \dots, n_j, j = 0, \dots, l\}$ is a Jordan basis for the operator $A|_{\mathcal{K}_0}$, we get $\alpha_{ij} = 0$ for $(i, j) \in J$.

The mapping

$$\mathcal{K}_0 \ni f \mapsto [f] \in \mathcal{S}_\lambda(A)/\mathcal{N}(A - \lambda)$$

is onto, so $\{[f_i^{(j)}] : (i, j) \in J\}$ is a basis for $\mathcal{S}_\lambda(A)/\mathcal{N}(A - \lambda)$.

The claim on the relation between Segre characteristics follows directly from the forms of basis. \square

Now we return to the case where the operator A under consideration is of the form $TT^{[\natural]}$ or $T^{[\natural]}T$. First we recall that the negative part of the spectrum of $T^{[\natural]}T$ is finite, thus there are no singular critical points on the negative part of the real axis. Moreover, all the algebraic root spaces corresponding to negative eigenvalues are finite dimensional (see, e.g., [15]).

Next let T be a closed densely defined operator in a Pontryagin space and let $\lambda \in \mathbb{C}$. Note that for $j = 1, \dots, 2\kappa + 1$ we have

$$T(\mathcal{N}((T^{[\natural]}T - \lambda)^j)) \subseteq \mathcal{N}((TT^{[\natural]} - \lambda)^j), \quad T^{[\natural]}(\mathcal{N}((TT^{[\natural]} - \lambda)^j)) \subseteq \mathcal{N}((T^{[\natural]}T - \lambda)^j). \quad (7.5)$$

In particular, the following operators are well defined:

$$[T] : \mathcal{S}_\lambda(T^{[\natural]}T)/\mathcal{N}(T^{[\natural]}T - \lambda) \ni [f] \mapsto [Tf] \in \mathcal{S}_\lambda(TT^{[\natural]})/\mathcal{N}(TT^{[\natural]} - \lambda)$$

$$[T^{[\natural]}] : \mathcal{S}_\lambda(TT^{[\natural]})/\mathcal{N}(TT^{[\natural]} - \lambda) \ni [f] \mapsto [T^{[\natural]}f] \in \mathcal{S}_\lambda(T^{[\natural]}T)/\mathcal{N}(T^{[\natural]}T - \lambda)$$

($[g]$ stands for the equivalence class of an element g both in $\mathcal{S}_\lambda(TT^{[\natural]})/\mathcal{N}(TT^{[\natural]} - \lambda)$ and $\mathcal{S}_\lambda(T^{[\natural]}T)/\mathcal{N}(T^{[\natural]}T - \lambda)$, the subscript λ has been omitted as before). Moreover,

$$[T^{[\natural]}T] = [T^{[\natural]}][T], \quad [TT^{[\natural]}] = [T][T^{[\natural]}]. \quad (7.6)$$

Theorem 7.2. *Let λ be a complex number, and denote by $(n_j)_{j=1}^\infty$ and $(m_j)_{j=1}^\infty$ the Segre characteristics for $T^{[\natural]}T$ and $TT^{[\natural]}$ respectively, corresponding to λ . If $\lambda \neq 0$ then $n_j = m_j$ for all $j \in \mathbb{N}$. If $\lambda = 0$ then $|n_j - m_j| \leq 1$ for $j \in \mathbb{N}$.*

Proof. First let $\lambda = 0$. The result of Flanders [6, Theorem 2], applied to the operators $[T^{[\natural]}][T]$ and $[T][T^{[\natural]}]$, together with (7.6) and Lemma 7.1 give

$$|\tilde{n}_j - \tilde{m}_j| \leq 1, \quad j \in \mathbb{N}, \quad (7.7)$$

where $(\tilde{n}_j)_{j=0}^\infty$ and $(\tilde{m}_j)_{j=0}^\infty$ are Segre characteristics for $[T^{[\mathfrak{h}]}T]$ and $[TT^{[\mathfrak{h}]}]$, respectively. The formulas (7.7) and (7.4) show that $|n_l - m_l| \leq 1$ for all l such that $n_l \geq 3$ or $m_l \geq 3$. Consequently, $|n_l - m_l| \leq 1$ also if $n_l = 1$ or if $m_l = 1$. Hence, the only case we have to exclude is $n_l = 2$, $m_l = 0$ (or conversely) for some $l \in \mathbb{N}$. Let then $n_l = 2$. Take the vectors $f_1^{(0)}, \dots, f_1^{(l)}$ from Lemma 7.1, which is possible because $n_j \geq n_l \geq 2$ for $j = 0, \dots, l$. Note that the vectors $Tf_1^{(0)}, \dots, Tf_1^{(l)}$ are linearly independent. Indeed, if

$$\sum_{j=0}^l \alpha_j Tf_1^{(j)} = 0 \quad \text{for some } \alpha_1, \dots, \alpha_l \in \mathbb{C},$$

then

$$\sum_{j=0}^l \alpha_j f_1^{(j)} \in \mathcal{N}(T) \subseteq \mathcal{N}(T^{[\mathfrak{h}]}T).$$

Since the vectors $[f_i^{(j)}]$, $j = 1, \dots, l$ are linearly independent (Lemma 7.1), we get $\alpha_j = 0$ ($j = 0, \dots, l$). Hence, there are l linearly independent vectors in $T(\mathcal{N}(T^{[\mathfrak{h}]}T)) \subseteq \mathcal{N}(TT^{[\mathfrak{h}]})$. Consequently $m_l \geq 1$ (otherwise there are at most $l - 1$ linearly independent vectors in $\mathcal{N}(TT^{[\mathfrak{h}]})$).

The case $\lambda \neq 0$ is similar. Indeed, in this case we can follow the same argument applying Theorem 1 in [6] instead of Theorem 2, to obtain that $\tilde{n}_j = \tilde{m}_j$ for $j \in \mathbb{N}$. As a result, $n_j = m_j$ whenever either one is larger than one. It remains to exclude the case $n_l = 1$ and $m_l = 0$ (or conversely) for some $l \in \mathbb{N}$. It should be noted that T maps $\mathcal{N}(T^{[\mathfrak{h}]}T - \lambda)$ in a one-to-one way onto $\mathcal{N}(TT^{[\mathfrak{h}]} - \lambda)$ ($\lambda^{-1}T^{[\mathfrak{h}]}$ is the inverse). Now suppose that $m_l = 0$, then it follows that $\mathcal{N}(TT^{[\mathfrak{h}]} - \lambda)$ is finite dimensional and by the observation in the previous sentence it follows that also $\mathcal{N}(T^{[\mathfrak{h}]}T - \lambda)$ is finite dimensional and has the same dimension. Hence, m_l must be zero as well. \square

8. Final remarks

The condition given by Flanders is necessary and sufficient. Namely, given two sequences $(n_j)_{j=0}^\infty$ and $(m_j)_{j=0}^\infty$ satisfying (1.1) we can always construct matrices A and B such that AB and BA have only zero in the spectrum and $(n_j)_{j=0}^\infty$ and $(m_j)_{j=0}^\infty$ are the Segre characteristics for AB and BA . This solves the finite dimensional problem completely. For the pair of operators $T^{[\mathfrak{h}]}T$ and $TT^{[\mathfrak{h}]}$ in a finite dimensional Pontryagin space this result is not true. For example it is easy to show that it is not possible that both operators $T^{[\mathfrak{h}]}T$ and $TT^{[\mathfrak{h}]}$ have only one Jordan chain of the same length bigger than one. Some parts of the analysis were done in [12], while a different perspective is taken in [16]. The latter paper solves the finite dimensional problem completely in different terms than ours. The reduction of the Pontryagin space case to the finite dimensional case involves, among other

things, the procedure described in the last section. We shall return to this issue in a subsequent paper.

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